

# Towards a Cosmology with Minimal Length and Maximal Energy

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## Abstract

The Friedmann-Robertson-Walker (FRW) universe and Bianchi I,II universes are investigated in the framework of the generalized uncertainty principle (GUP) which predicts minimum measurable length as well as maximum measurable momentum. We get a dynamic cosmological bounce for the FRW universe. With Bianchi universe, we found that the universe may be still isotropic by implementing GUP. Moreover, the wall velocity appears to be stationary with respect to the universe velocity which means that when the momentum of the Universe evolves into a maximum measurable energy, the bounce is enhanced against the wall which means no maximum limit angle is manifested anymore.

## 1 Introduction

The existence of space-like singularities is a generic feature of Einstein's general theory of relativity and this is proved by the singularity theorems of Penrose, Hawking and Geroch [1]. The singularities are usually characterized by divergences of curvature invariants or breakdown of geodesics. This indicates a limit beyond which general theory relativity is not applicable anymore. It is widely believed that a theory of quantum gravity will be the one which can provide insights on the resolution of singularities. In fact there are different approaches for quantum theory of gravity, so the description at low energy limit also have competing candidates even at the phenomenological level.

Various approaches for quantum gravity such as string theory and black hole physics, have predicted the existence of a minimum measurable length and an essential modification of the Heisenberg uncertainty principle [2–7] to the so-called generalized uncertainty principle (GUP). The GUP is based on the modification of the fundamental commutation relation mainly in position and momenta. Recent developments suggests that the minimal length scale can be implemented in quantum mechanics and quantum field theory through the GUP. Within the framework of the GUP minimal length scale has been studied in details in quantum mechanics, quantum electrodynamics, thermodynamics, black-hole physics and cosmology. For some recent reviews on the phenomenology of different minimal length scale scenarios inspired by various approaches to the quantum gravity see [8].

Quite recently Ali et al. [10, 11] introduced a new approach which predicts maximum observable momenta besides a minimal measurable length. This model is built to be consistent with DSR [9], string theory and black hole physics [2, 3]. Furthermore, it ensures that  $[x_i, x_j] = 0 = [p_i, p_j]$  through Jacobi identity [12]. Accordingly, the modification of the Heisenberg uncertainty relation near the Planck scale reads

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$$[x_i, p_j] = i\hbar \left[ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j) \right]. \quad (1.1)$$

Here  $\alpha$  is the deformation parameter given by  $\alpha = \alpha_0/(c M_{pl}) = \alpha_0 l_{pl}/\hbar$ .  $c$ ,  $M_{pl}$  and  $l_{pl}$  are speed of light in vacuum, Planck mass and Planck length respectively.  $\alpha_0$  is a dimensionless parameter usually considered to be of order unity. The upper bounds on the parameter  $\alpha_0$  has been calculated in [12] and it was predicted that it could predict an intermediate length scale between Planck scale and electroweak scale. A recent proposal suggested that these bounds can be measured using quantum optics techniques in [13,14] which may be considered as a milestone in quantum gravity phenomenology. Due to the GUP as proposed in [10] the physical momentum is redefined. As a result the classical Hamiltonian as well as the quantum Hamiltonian gets modified which affects the quantum phenomena. Recently, Bekenstein [15, 16] proposed that quantum gravitational effects could be tested experimentally, he suggests “a tabletop experiment which, given state of the art ultrahigh vacuum and cryogenic technology, could already be sensitive enough to detect Planck scale signals” [15]. This would put several quantum gravity predictions to test in the Laboratory [13,14]. Definitely, this is considered as a milestone in the road of quantum gravity phenomenology. So it is important to make a quantitative study of these effects which would open an interesting window for quantum gravity phenomenology especially if the GUP parameter  $\alpha_0$  lies between the Planck scale and the electroweak scale. In a series of papers, the effects of GUP on atomic systems, condensed matter systems, preheating phase of the universe, inflationary era of the universe and black holes have been investigated [12,17–30].

In this paragraph we shed some light on the phase space formulation of quantum mechanics as in the later sections we are going to use GUP modified classical Poisson brackets. The known correspondence between commutator and Poisson bracket was first proposed by Dirac where he proposed that the quantum counterparts  $\hat{A}$ ,  $\hat{B}$  of classical observables  $A$ ,  $B$  satisfy [31]

$$[\hat{A}, \hat{B}] = i\hbar\{A, B\} \quad (1.2)$$

and since that time, this is known as a postulate of Quantum Mechanics. In 1946, Hip Groenewold demonstrated that a general systematic correspondence between quantum commutators and Poisson brackets could not hold consistently [32]. However, he did appreciate that such a systematic correspondence does, in fact, exist between the quantum commutator and a deformation of the Poisson bracket, which is called the Moyal bracket [33]. The Moyal bracket is a way of describing the commutator of observables in the phase space formulation of quantum mechanics when these observables are described as functions on phase space. It relies on schemes for identifying functions on phase space with quantum observables, the most famous of these schemes being Weyl quantization [34]. It underlies Moyal’s dynamical equation, an equivalent formulation of Heisenberg’s quantum equation of motion, thereby providing the quantum generalization of Hamilton’s equations. In a two-dimensional flat phase space, and for the Weyl-map correspondence, the Moyal bracket reads,

$$\{\{f, g\}\} = \frac{1}{i\hbar}(f \star g - g \star f) = \{f, g\} + \mathcal{O}(\hbar^2). \quad (1.3)$$

Where  $\star$  is the star-product operator in phase space (Moyal product), while  $f$  and  $g$  are differentiable phase-space functions, and  $\{f, g\}$  is their Poisson bracket. This means Moyal bracket

is equal to poisson bracket in its equivalence with the quantum commutator up to the second order of  $\hbar$  (i.e  $\hbar^2$ ). Fortunately, in our GUP model, the correction which is proportional to  $\alpha$ , and for which we do our calculations in our whole paper, in the commutator of Eq. (1.1) is of order  $\hbar\alpha \sim \alpha_0\ell_P \sim \hbar^{1/2}$  where  $\ell_P = \sqrt{\hbar G/c^3}$ , which means Moyal bracket is not necessary in our model and Poisson bracket is quite enough to study the commutator of GUP as a Poisson bracket according to the postulate of Quantum Mechanics that has been proposed by Dirac. In quantum mechanics, the coordinates  $p$  and  $q$  of phase space normally become hermitian operators in a Hilbert space. But they may alternatively retain their classical interpretation, provided functions of them composed in novel algebraic ways (through Groenewold's 1946 star product [32]), consistent with the uncertainty principle of quantum mechanics. Every quantum mechanical observable corresponds to a unique function or distribution on phase space, and vice versa, as specified by Hermann Weyl (1927) and supplemented by John von Neumann (1931); Eugene Wigner (1932); and, in a grand synthesis, by H J Groenewold (1946). With J E Moyal (1949) [33], these completed the foundations of the phase space formulation of quantum mechanics [35]. Thus, by expressing quantum mechanics in phase space (the same ambit as for classical mechanics), the Weyl map facilitates recognition of quantum mechanics as a deformation (generalization) of classical mechanics, with deformation parameter  $\hbar/S$ , where  $S$  is the action of the relevant process. Classical expressions, observables, and operations (such as Poisson brackets) are modified by  $\hbar$ -dependent quantum corrections, as the conventional commutative multiplication applying in classical mechanics is generalized to the noncommutative star-multiplication characterizing quantum mechanics and underlying its uncertainty principle [35].

Motivated by the GUP, many authors studied some classical problems with deformed Poisson brackets [36,37]. From the phenomenological point of view, constraints were placed on deformation parameters in [36] by considering the effects of GUP motivated deformed Poisson algebra on the classical orbits of particles in the central force problem. In [37] it was conjectured that modified commutation relations as suggested by candidate theories of quantum gravity, persist in the classical limit also. There the perihelion precession rate for Keplerian orbits was calculated. A deformed Heisenberg algebra or in the classical case, a deformed Poisson algebra incorporate some additional problems like the violation of equivalence principle [22]. But recently in [38] it was shown that the GUP is reconciled with the equivalence principle. The effects of the GUP on Galilean and Lorentz transformation is also studied recently in [39].

The scope of the present work is to investigate the effect of the GUP on the dynamics of the Friedmann-Robertson-Walker universe (FRW) by performing this modification at the classical level by studying the modifications induced on the symplectic geometry by the deformed algebra. It is found that big bang singularity seems to be suppressed, by considering GUP. It is found that modified Friedmann equation predicts a cosmological bounce. Besides, we extend our study to investigate the effect of GUP on Bianchi I and II universes. In each case the deformed commutation relations due to GUP would modify the dynamical equations and hence we will find departures from the usual scenario. We investigate the isotropy of Bianchi universes as well as the singularity. An outline of this paper is as follows: in Sec. 2, we investigate the GUP with FRW universe and derive the modified Friedmann equation. In Sec. 3, we tackle the impact of GUP on Bianchi universes with type I and II. We compare our results with previous studies in [40].

## 2 FRW universe in the framework of GUP

Here we study the GUP deformed dynamics of an isotropic and homogeneous cosmological model. Let us start with a review of the standard case. The FRW metric can be described by the line element as

$$ds^2 = -N^2 dt^2 + a(t)^2 \left( \frac{dr^2}{1 - k r^2} + r^2 d\Omega^2 \right) \quad (2.1)$$

where  $a(t)$  is the scale factor of the universe and  $N$  is the lapse function. The constant  $k$  is a measure of spatial flatness and it can be  $0, \pm 1$ . The dynamics of such models are summarized in the Hamiltonian constraint

$$\mathcal{H} = -\frac{2\pi G}{3} \frac{p_a^2}{a} - \frac{3}{8\pi G} a k + a^3 \rho = 0 \quad (2.2)$$

where  $G$  is the gravitational constant and  $\rho$  represents the matter density in the universe. The Freidmann equation can be derived using Poisson brackets

$$\{A, B\} = \left( \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_j} \right) \{x_i, p_j\} + \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} \{x_i, x_j\} \quad (2.3)$$

where  $x_i$  and  $p_j$  are the canonical variables of the system. In the standard case, the canonical variables are given by  $a$  and  $p_a$  and these canonical variables satisfy the Poisson bracket as  $\{a, p_a\} = 1$ . This is due to the fact that isotropy reduces the phase space of the model to two dimensional. The Friedmann equations can be extracted by the Hamilton's equations which can be derived from the extended Hamiltonian

$$\mathcal{H}_E = \frac{2\pi G}{3} \frac{N p_a^2}{a} + \frac{3}{8\pi G} N a k - N a^3 \rho + \lambda \mathcal{P} \quad (2.4)$$

where  $\lambda$  is the Lagrange multiplier, and  $\mathcal{P}$  is the momentum conjugate to the lapse  $N$  which vanishes. This would give the equation of motion for the lapse  $\dot{N} = \{N, \mathcal{H}_E\} = \lambda$  and the scalar constraint of Eq. (2.2) is obtained by the fact that the constraint  $\mathcal{P} = 0$  will be satisfied at all times which is equivalent to demand that the secondary constraint  $\dot{\mathcal{P}} = \{\mathcal{P}, \mathcal{H}_E\} = \mathcal{H} = 0$  holds. The other equations of motion regarding  $a$  and  $p_a$  with respect to extended Hamiltonian are

$$\dot{a} = \{a, \mathcal{H}_E\} = \frac{\partial \mathcal{H}_E}{\partial p_a} = \frac{4\pi G}{3} N \frac{p_a}{a} \quad (2.5)$$

$$\dot{p}_a = \{p_a, \mathcal{H}_E\} = N \left( \frac{2\pi G}{3} \frac{p_a^2}{a^2} - \frac{3}{8\pi G} k + 3a^2 \rho + a^3 \frac{d\rho}{da} \right) \quad (2.6)$$

By using the equations (2.5) and (2.6) and the scalar constraint (2.2), we can obtain the equation of motion for  $\dot{a}$  which represents the Freidmann equation

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad (2.7)$$

Usually  $\dot{a}/a$  is called the Hubble rate. It is well known that this classical equation breaks down at the big-bang singularity and a natural crisis for a quantum description of the Universe in Planck scale occurs.

Now we use the GUP as proposed in [10, 11] for a heuristic analysis of the singularity at big-bang. Here we intend to study the classical equations only. As discussed earlier in details, here we will consider only the GUP motivated deformed Poisson algebra to study the classical equations that govern our universe. With implementing the GUP up to the first order of  $\alpha$ , we get the Poisson bracket between  $a$  and  $p$  as

$$\{a, p\} = 1 - 2\alpha p \quad . \quad (2.8)$$

Here we have replaced the quantum mechanical commutator by the Poisson bracket. So, using the Poisson algebra of Eq. (2.3), we get the modified equations of motions for FRW universe as

$$\dot{a} = \{a, \mathcal{H}_E\} = \frac{\partial H_E}{\partial p_a} (1 - 2\alpha p_a) \quad (2.9)$$

$$\dot{p}_a = \{p_a, \mathcal{H}_E\} = -\frac{\partial H_E}{\partial a} (1 - 2\alpha p_a) \quad (2.10)$$

Using the expression for the extended Hamiltonian we follow the same procedure as in the standard case and get the modified equations of motions as

$$\dot{a} = \frac{4\pi G}{3} N \frac{p_a}{a} (1 - 2\alpha p_a) \quad (2.11)$$

$$\dot{p}_a = \{p_a, H_E\} = N \left( \frac{2\pi G}{3} \frac{p_a^2}{a^2} - \frac{3}{8\pi G} k + 3a^2 \rho + a^3 \frac{d\rho}{da} \right) (1 - 2\alpha p_a) \quad (2.12)$$

Using equations (2.11) and (2.12) with the scalar constraint (2.2), we obtain the equation of motion for the Hubble rate  $\dot{a}/a$  which represents the modified/deformed Friedmann equation due to GUP as

$$\left( \frac{\dot{a}}{a} \right)^2 = \left( \frac{8\pi G}{3} \rho - \frac{k}{a^2} \right) \left[ 1 - 2\alpha a^2 \sqrt{\frac{3}{2\pi G}} \left( \rho - \frac{3}{8\pi G} \frac{k}{a^2} \right)^{1/2} \right] \quad (2.13)$$

By considering the spatially flat case in which  $k=0$ , we find that the modified Friedmann equation will be

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho \left[ 1 - 2\alpha a^2 \sqrt{\frac{3}{2\pi G}} \rho^{1/2} \right] \quad (2.14)$$

This equation appears to represent a big bounce picture for the FRW universe with some critical density. This equation appears to introduce an interesting result, where it assumes that there is a connection between the matter density  $\rho$  and the expansion rate  $a$ . It implies that the value of matter density  $\rho$  is always bounded with  $1/(2\alpha a^2 \sqrt{\frac{3}{2\pi G}})$  so the correction due to GUP is always kept  $\leq 1$ , or otherwise we will get an imaginary density or expansion rate. This behavior is quite similar to the effect of GUP on black hole thermodynamics, in which the final stages of a black hole reach to a remnant due to GUP effect, and this remnant does not radiate any more, otherwise, we will get imaginary Hawking temperature [41, 42]. This lead us to define a critical density in the considered model as follows:

$$\rho_c^{1/2} = \frac{1}{2\alpha a^2 \sqrt{\frac{3}{2\pi G}}} \quad (2.15)$$

The inbuilt minimal length in the theory restricts the limit  $a \rightarrow 0$  which keeps the *dynamical*<sup>3</sup> value of  $\rho_c$  finite. Our modified Friedmann equation (2.14) has a very close resemblance with results found in [43]. In [43] the modified Friedmann equations were studied with the form of the generalized uncertainty principle deduced from the Snyder non-commutative space. There it was shown that the resulting cosmological model predicts a cosmological bounce at some critical energy density.

### 3 Bianchi Universes and GUP

We now study the modified equations of motion for the Bianchi Universe by implementing the minimal length in Quantum gravity which assume a natural cut-off on the anisotropies. The Bianchi Universes are spatially homogeneous cosmological spacetimes and the symmetry group acts on each spatial manifold [44]. In the Misner formalism [45] the Hamiltonian constraint governing the dynamics can be written as

$$H = -p_\gamma^2 + p_+^2 + p_-^2 + e^{4\gamma}V(x_\pm) = 0 \quad , \quad (3.1)$$

where the lapse function  $N = -e^{3\gamma}/2p_\gamma$  is fixed by the time gauge  $\dot{\gamma} = 1$ .  $\gamma$  describes the isotropic expansion and the anisotropies are described by  $x_\pm$ . The classical singularity occurs in the limit  $\gamma \rightarrow -\infty$  and the potential term  $V(x_\pm)$  makes the classification between the models which is associated with the scalar curvature. Now it is necessary to make a choice of the time parameter for analyzing the dynamics of the system. Since the volume of the universe is proportional to  $e^{3\gamma}$  so here we consider  $\gamma$  to be the time parameter and obtain an effective Hamiltonian by solving the constraint equation (3.1) with respect to  $p_\gamma$ . The effective Hamiltonian can be written as

$$\mathcal{H} = -p_\gamma = [p_+^2 + p_-^2 + e^{4\gamma}V(x_\pm)]^{\frac{1}{2}} \quad (3.2)$$

Let us now consider the modifications on the phase space as introduced by the GUP. Here we will use the GUP of [10] for our purpose and according to it the Poisson brackets are

$$\{x_i, p_i\} = \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j), \quad (3.3)$$

$$\{p_i, p_j\} = \{x_i, x_j\} = 0. \quad (3.4)$$

For any phase space function we have

$$\{A, B\} = \left( \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_j} \right) \{x_i, p_j\} + \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} \{x_i, x_j\} \quad (3.5)$$

Here we have considered the requirements that the GUP deformed Poisson brackets must also be bilinear, anti-symmetric, following the Leibniz rules and the Jacobi identity. The deformed classical dynamics of the Bianchi models can be obtained from the phase algebra of Eqs. (3.3, 3.4, 3.5). We derive below the time dependence of anisotropies and their conjugate momenta using the effective Hamiltonian of Eq. (3.2) as

$$\{x_i, \mathcal{H}\} = \dot{x}_i = \frac{p_k}{H} \left( \delta_{ik} - \alpha(p\delta_{ij} + \frac{p_i p_j}{p}) + \alpha^2(p^2 \delta_{ik} + 3p_i p_j) \right) \quad , \quad (3.6)$$

$$\{p_i, \mathcal{H}\} = \dot{p}_i = -\frac{1}{2\mathcal{H}} e^{4\gamma} \left( \delta_{ik} - \alpha(p\delta_{ij} + \frac{p_i p_j}{p}) + \alpha^2(p^2 \delta_{ik} + 3p_i p_j) \right) \frac{\partial V}{\partial x_k} \quad , \quad (3.7)$$

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<sup>3</sup>By dynamical we mean the dependence of  $\rho_c$  on the scale factor  $a$ .

the differentiation in the last two equations is in terms of the time variable  $\gamma$  and  $p^2$  is defined as  $p^2 = p_+^2 + p_-^2$ . The equations (3.7, 3.6) represent the modified equations of motion for the homogeneous Universes due to GUP. In the next subsections, we are going to study in details the Bianchi I and II universes based on this formalism.

### 3.1 Bianchi I in the framework of GUP

The main property of Bianchi I universe is to be homogeneous and it has flat spatial sections [44,46]. Bianchi I corresponds to the case  $V(x_{\pm}) = 0$  in the scheme of Eq. (3.2). This implies that the solution of the equations of motion (3.6, 3.7) of Bianchi I Universe is Kasner-like. Here we have

$$\dot{p}_i = 0 \quad (3.8)$$

$$\begin{aligned} \dot{x}^2 &= \frac{p^2}{\mathcal{H}^2} [1 - 4\alpha p + 12\alpha^2 p^2 - 16\alpha^3 p^3 + 16\alpha^4 p^4] \\ &= [1 - 4\alpha p + 12\alpha^2 p^2 - 16\alpha^3 p^3 + 16\alpha^4 p^4] \\ &= [1 - 2\alpha p + 4\alpha^2 p^2]^2 \end{aligned} \quad (3.9)$$

As  $\alpha = 0$ , we get standard Kasner velocity which is  $\dot{x}^2 = 1$ . The effect of GUP and its cutoff enhance the values Kasner velocity. In the last step, we used the fact for the Bianchi I universe, the ADM Hamiltonian is constant and given by  $\mathcal{H}^2 = p^2 = \text{const.}$

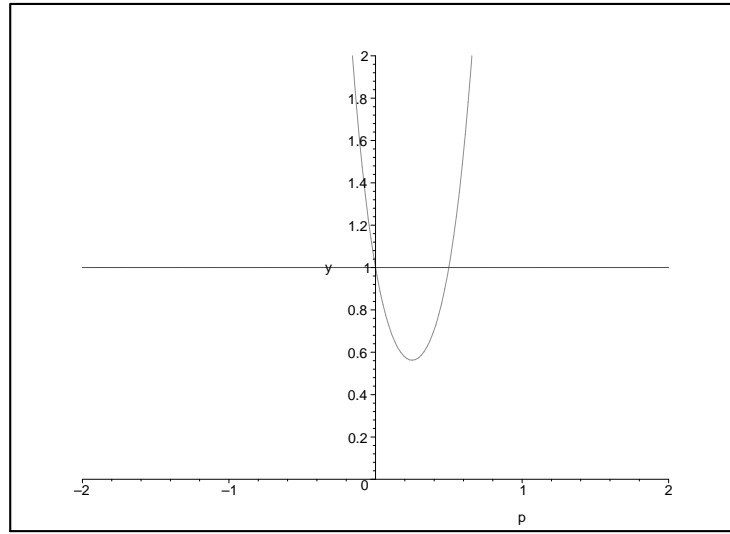


Figure 1:  $\dot{x}^2$  versus  $p$  for standard and GUP modified cases

We investigate here how the Kasner behavior is affected by the GUP deformed framework. The spatial metric of the Kasner solution is written as

$$dl^2 = t^{2s_1} dx_1^2 + t^{2s_2} dx_2^2 + t^{2s_3} dx_3^2, \quad (3.10)$$

where  $s_1, s_2, s_3$  are the Kasner indices which follow the following equations:

$$s_1^2 + s_2^2 + s_3^2 = 1 \quad (3.11)$$

$$s_1 + s_2 + s_3 = 1. \quad (3.12)$$

The first Kasner relation is directly related to the anisotropy velocity  $\dot{x}$  by the equations [46]

$$\dot{x}_+ = \frac{1}{2}(1 - 3s_3) \quad (3.13)$$

$$\dot{x}_- = \frac{\sqrt{3}}{2}(s_1 - s_2) \quad (3.14)$$

and the second arises from the arbitrariness in choosing the tetrads and is still valid in the GUP deformed formulation. Then, the first Kasner relation is then deformed as

$$\begin{aligned} s_1^2 + s_2^2 + s_3^2 &= 1 - 4\alpha p + 12\alpha^2 p^2 - 16\alpha^3 p^3 + 16\alpha^4 p^4 \\ &= 1 - 4\sqrt{\mu} + 12\mu - 16\mu^{3/2} + 16\mu^2 \quad (3.15) \end{aligned}$$

Here we have defined  $\mu = \alpha^2 p^2$  as a measure of deformation in terms of the anisotropy momentum.  $\mu = 0$  gives the standard result. Now it is very important to note that in the GUP framework that we are using the terms on the right hand side of Eq. (3.15) have alternating signs. This means that the Universe can isotropize and can reach a situation when the Kasner indices become equal. But in the framework of GUP as introduced by Kempf et. al. [4] the Universe cannot isotropize and the Kasner dynamics is highly modified which is just the opposite as in our case [40]. So in our case we can accommodate the contraction along two directions while approaching the classical singularity and is similar to what happens in the standard case. To capture the GUP effects in detail we need to investigate the GUP deformed Bianchi II Universe.

### 3.2 Bianchi II in the framework of GUP

In this section we will study Bianchi II dynamics in the same framework of the GUP. Bianchi II connects the homogeneous flat Universe of that of Bianchi I with Bianchi IX. Bianchi II corresponds to Bianchi IX when we consider only one potential wall [44, 46]. As we are considering the Hamiltonian analysis, for Bianchi II the potential is as follow;  $V(x_{\pm}) = e^{-8x_{\pm}}$ . We can write the Hamiltonian as

$$\mathcal{H} = [p_+^2 + p_-^2 + e^{4(\gamma - 2x_+)}]^{1/2} \quad (3.16)$$

The main difference of the GUP framework with respect to the standard one is that  $\mathcal{H}$  is not anymore a constant of motion in the vicinity of the classical singularity  $\gamma \rightarrow -\infty$ . We need to investigate the bounce of the Universe (or analogous to particle) against the potential wall in the GUP framework. For that we need to have the expression of wall velocity. In this case the wall velocity is written as follows [40, 46]

$$\dot{x}_+ \approx \dot{x}_w = \frac{1}{2} - \frac{1}{8} \frac{\partial}{\partial \gamma} \ln \mathcal{H}^2 \quad (3.17)$$

Here in the GUP framework  $\mathcal{H}$  is not a constant and we can write

$$\frac{\partial}{\partial \gamma} \ln \mathcal{H}^2 = 4 \left[ 1 - \frac{p^2(\dot{x})}{\mathcal{H}^2} \right] \quad (3.18)$$

So the wall velocity now becomes

$$\begin{aligned} \dot{x}_w &= \frac{p^2(\dot{x})}{2\mathcal{H}^2} = \frac{\dot{x}^2}{2} [1 - 4\alpha p + 12\alpha^2 p^2 - 16\alpha^3 p^3 + 16\alpha^4 p^4]^{-1} \\ &= \frac{\dot{x}^2}{2} [1 - 2\alpha p + 4\alpha^2 p^2]^{-2} \quad (3.19) \end{aligned}$$



Here we have used Eq. (3.9) for  $\dot{x}$  and considered that  $\mathcal{H}^2 = p^2$  near the classical cosmological singularity ( $\gamma \rightarrow -\infty$ ). Eq. (3.19) can further be written as

$$\frac{\dot{x}_w}{\dot{x}} = \frac{1}{2}(1 - 2\sqrt{\mu} + 4\mu)^{-1} , \quad (3.20)$$

where we have defined  $\mu = \alpha^2 p^2$ . In the undeformed state  $\mu = 0$ , we recover the standard picture where  $\dot{x} = 1$  and  $\dot{x}_w = 1/2$ . In Fig. (2) we plot the ratio of wall velocity and particle velocity as a function of  $\mu$ . We can see that in the GUP framework that we are studying, the ratio  $\dot{x}_w/\dot{x}$  is higher and have a maximum for low values of  $\mu$  or in some sense in the quasi standard regime. The result converges with the result of the GUP framework of [4] for high values of  $\mu$  or in the highly deformed regime. In our case we can clearly see that the bounce is accelerated for low values of  $\mu$  as there is maximum of  $\dot{x}_w/\dot{x}$  at around 0.67.

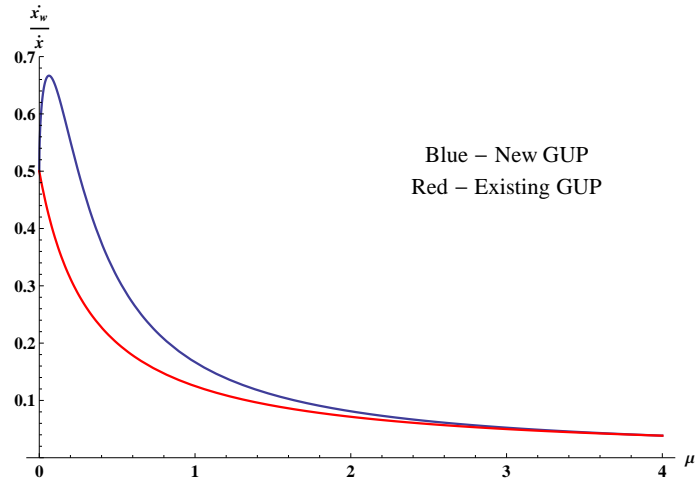


Figure 2: Plot of  $\dot{x}_w/\dot{x}$  as a function of  $\mu$ . Here we have compared our result (blue) with the result of the GUP as introduced in [4] (red).

In the GUP framework the particle (Universe) as well as the wall velocity depends on the anisotropy momentum and the deformation parameter  $\alpha$ . Here  $\dot{x}_w/\dot{x} \neq 1/2$  as what is expected in the standard case. In the deformed case in the asymptotic limit of very large  $\mu$ ,  $\dot{x}_w/\dot{x}$  vanishes and the wall appears stationary with respect to the particle (Universe) velocity. The maximum angle for the bounce to happen is evaluated as  $|\theta_{max}| = \pi/3$  in the standard case [47]. In the asymptotically limit of the highly deformed case ( $\mu \gg 1$ ) the maximum angle is evaluated as  $|\theta_i| < |\theta_{max}| = \cos^{-1}(\dot{x}_w/\dot{x}) = \pi/2$ . Based on this, when the momentum of the particle (Universe) evolve into a maximum measurable energy, the bounce is enhanced against the wall which means no maximum limit angle is manifested anymore. Also we have a maximum of  $\dot{x}_w/\dot{x}$  for low values of  $\mu$  at around 0.67 where the phenomenon of bounce is accelerated. The maximum angle for this case is  $|\theta_{max}| \approx \frac{4\pi}{15}$ .

## 4 Conclusions

In this work we have investigated the consequences of the GUP, which predicts maximum observable momenta besides a minimal measurable length [10, 11] and is consistent with DSR [9], string theory and black hole physics [2, 3], on FRW universe and Bianchi I and II universes. By investigating the effect of GUP with FRW universe, we found that the big

bang singularity seems to be suppressed by a dynamical critical density (dynamical in the sense that it is scale factor dependent). Moreover, we extend our study for Bianchi I and II universes. By investigating GUP with Bianchi I, we found that the Universe may possibly be isotropic and may evolve into a situation at which the Kasner indices become equal with implementing GUP. With Bianchi II, we found that in the asymptotic limit of the highly deformed case ( $\mu \gg 1$ ) the maximum angle is given by  $|\theta_i| < |\theta_{max}| = \cos^{-1}(\dot{x}_w/\dot{x}) = \pi/2$ . when the momentum of the particle (Universe) evolves into a maximum measurable energy, the bounce is enhanced against the wall which means no maximum limit angle is manifested anymore. For low  $\mu$ , we have a maximum of  $\dot{x}_w/\dot{x}$  for low values of  $\mu$  at around 0.67 where the phenomenon of bounce is accelerated. The maximum angle for this case is  $|\theta_{max}| \approx \frac{4\pi}{15}$ . We conclude that the proposed GUP in this work might possibly resolve singularity problems with the considered universes and may imply a bounce picture for the universe.

We should also point out that an expanding universe becomes isotropic due to the matter contributions. But here we have not considered any particular matter which is a shortcoming of our present model. So it is indeed necessary to study the effective equations of our model with different matter contributions which we plan to consider in near future.

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